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INTEGRATION OF POLYNOMIALS OVER AN ARBITRARY TETRAHEDRON IN EUCLIDEAN THREE-DIMENSIONAL SPACE

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Abstract—In this paper, we present explicit integration formulas and algorithms for computing integrals of polynomials over an arbitrary tetrahedron in Euclidean three-dimensional space. Two different approaches are discussed: the first algorithm/formula is obtained by mapping the arbitrary tetrahedron into a unit orthogonal tetrahedron, while the second algorithm/formula computes the required integral as a sum of four integrals over the unit triangle. These algorithms/formulas are followed by an example for which we have explained the detailed computational scheme. The numerical result thus found is in complete agreement with the previous work. Further, it is shown that the present algorithms are much simpler and more economical as well in terms of arithmetic operations.

INTRODUCTION

Volume, center of mass, moment of inertia and other geometric properties of rigid homogeneous solids frequently arise in a large number of engineering applications, in CAD/CAE/CAM applications in geometric modelling as well as in robotics. Integration formulas for multiple integrals have always been of great interest in computer applications, a good overview of available methods for evaluating volume integrals is given by Lee and Requicha [1]. Timmer and Stern [2] discussed a theoretical approach to the evaluation of volume integrals by transforming the volume integral to a surface integral over the boundary of the integration domain. Lien and Kajiya [3] presented an outline of a closed form formula for volume integration by decomposing the solid into a set of solid tetrahedra. Cattani and Paoluzzi [4, 5] gave a symbolic solution to both the surface and volume integration of polynomials by using a triangulation of the solid boundary. In a recent paper, Bernardini [6] has presented explicit formulas and algorithms for computing integrals of polynomials over n-dimensional polyhedra by using the decomposition representation and the boundary representation of the polyhedron.

PROBLEM STATEMENT FOR PRESENT WORK

Most computational studies of triple integrals deal with problems in which the domain of integration is very simple, like a cube or a sphere, but the integrand is complicated. However, in real applications, we confront the inverse problem; the integrating function $f(x, y, z)$ is usually simple; but the domain is very complicated. Hence, in this paper and in other previous works [3–6] an attempt is made to obtain practical formulas for the exact evaluation of integrals

$$\iiint_P f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where P is a three-polyhedron in R^3 and $\mathrm{d}V$ is the differential volume. The integrating-function is a trivariate polynomial

$$f(x, y, z) = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{m} \sum_{\gamma=0}^{p} a_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma,$$

where α, β, γ_r are non-negative integers. However, the paper is focused on the calculation of the following integral of monomials:

$$III_V^{\alpha\beta\gamma} = \iiint_V x^\alpha y^\beta z^\gamma\,\mathrm{d}V,$$

where V is an arbitrary tetrahedron with four vertices (x_i, y_i, z_i) $(i = 1, 2, 3, 4)$ and an extension to $f(x, y, z)$ can be obtained by the linearity of integrals. Two different approaches are considered: the first algorithm is based on the fact that an arbitrary tetrahedron can always be transformed into an orthogonal unit tetrahedron by means of a mapping, the second algorithm is based on yet another well known theorem of Gauss (Gauss's divergence theorem) according to which the volume integrals may be reduced to surface integrals.

VOLUME INTEGRATION OVER AN ARBITRARY TETRAHEDRON

In this section, we first obtain the volume integral of a scalar function

$$f(p) = x^\alpha y^\beta z^\gamma$$

(α, β, γ, positive integers) over an arbitrary tetrahedron by transforming it to an orthogonal unit tetrahedron. That is we are actually interested in evaluating

$$III^{\alpha\beta\gamma} = \iiint_V x^\alpha y^\beta z^\gamma \, \mathrm{d}V, \tag{1}$$

where V is an arbitrary tetrahedron in the x, y, z Cartesian coordinate system.

Theorem 1

A structure product $III_V^{\alpha\beta\gamma}$ over the volume V of an arbitrary tetrahedron is a polynomial combination of the coordinates of vertices: (x_i, y_i, z_i), $(i = 1, 2, 3, 4)$

$$III_V^{\alpha\beta\gamma} = |\det J| \left[\tfrac{1}{6} x_4^\alpha y_4^\beta z_4^\gamma + \sum_{n=1}^{\alpha+\beta+\gamma} \times \sum_{n+n_2+n_3=n} G(n_1, n_2, n_3) III_{\bar{V}}^{n_1 n_2 n_3} \right] \tag{2}$$

where

$|\det J| = 6T$,

T = volume of tetrahedran with vertices,

$$(x_i, y_i, z_i) \quad (i = 1, 2, 3, 4) \tag{3}$$

$$\det J = \begin{vmatrix} (x_1 - x_4) & (x_2 - x_4) & (x_3 - x_4) \\ (y_1 - y_4) & (y_2 - y_4) & (y_3 - y_4) \\ (z_1 - z_4) & (z_2 - z_4) & (z_3 - z_4) \end{vmatrix} \tag{4}$$

$$G(n_1, n_2, n_3) = \frac{1}{\underline{/n_1}\, \underline{/n_2}\, \underline{/n_3}} \left(\frac{\partial^{n_1+n_2+n_3} f(\xi, \eta, \zeta)}{\partial \xi^{n_1} \partial \eta^{n_2} \partial \zeta^{n_3}} \right)_{(0,0,0)} \tag{5}$$

$$f(\xi, \eta, \zeta) = x^\alpha(\xi, \eta, \zeta) y^\beta(\xi, \eta, \zeta) z^\gamma(\xi, \eta, \zeta), \tag{6}$$

and $III_{\bar{V}}$ is the structure product

$$III_{\bar{V}} = \iiint_{\bar{V}} \xi^{n_1} \eta^{n_2} \zeta^{n_3} \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta = \frac{\underline{/n_1}\, \underline{/n_2}\, \underline{/n_3}}{(n_1 + n_2 + n_3 + 3)!}. \tag{7}$$

Over the unit orthogonal tetrahedron

$$\bar{V} = \langle (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 1, 0) \rangle$$

in the ξ, η, ζ Cartesian coordinate system.

Proof: the volume (natural) coordinates are related to Cartesian coordinates by the well-known relations [7]

$$x = L_i x_i, \quad y = L_i y_i, \quad z = L_i z_i$$

and

$$L_1 + L_2 + L_3 + L_4 = 1 \tag{8}$$

where (x_i, y_i, z_i) refer to the Cartesian coordinates of vertex i of the tetrahedron. Letting $L_1 = \xi_1$, $L_2 = \eta$, $L_3 = \zeta$, we can rewrite the relations (8) as:

$$x(\xi, \eta, \zeta) = x_4 + \xi x_{14} + \eta x_{24} + \zeta x_{34}$$

$$y(\xi, \eta, \zeta) = y_4 + \xi y_{14} + \eta y_{24} + \zeta y_{34}$$

$$z(\xi, \eta, \zeta) = z_4 + \xi z_{14} + \eta z_{24} + \zeta z_{34}$$

with

$$x_{ij} = x_i - x_j, \quad y_{ij} = y_i - y_j, \quad z_{ij} = z_i - z_j. \tag{9}$$

If we consider the mapping (see Fig. 1) between the three-dimensional space X, Y, Z and the three-dimensional space ξ, η, ζ by the parametric eqn (9), we have for the volume element

$$\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = |\det J| \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta \tag{10}$$

where

$$\det J = \begin{vmatrix} (x_1 - x_4) & (x_2 - x_4) & (x_3 - x_4) \\ (y_1 - y_4) & (y_2 - y_4) & (y_3 - y_4) \\ (z_1 - z_4) & (z_2 - z_4) & (z_3 - z_4) \end{vmatrix}$$

$$|\det J| = 6 \times \text{volume of tetrahedron} = \text{absolute value of } \det J. \tag{11}$$

So, if we change the coordinates according to eqn (9) and express consistently the volume element, we obtain

$$III_V^{\alpha\beta\gamma} = |\det J| \iiint_{\bar{V}} x^\alpha(\xi, \eta, \zeta) y^\beta(\xi, \eta, \zeta) \times z^\gamma(\xi, \eta, \zeta) \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta. \tag{12}$$

where $\bar{V}$ is the unit orthogonal tetrahedron

$$\langle (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) \rangle.$$

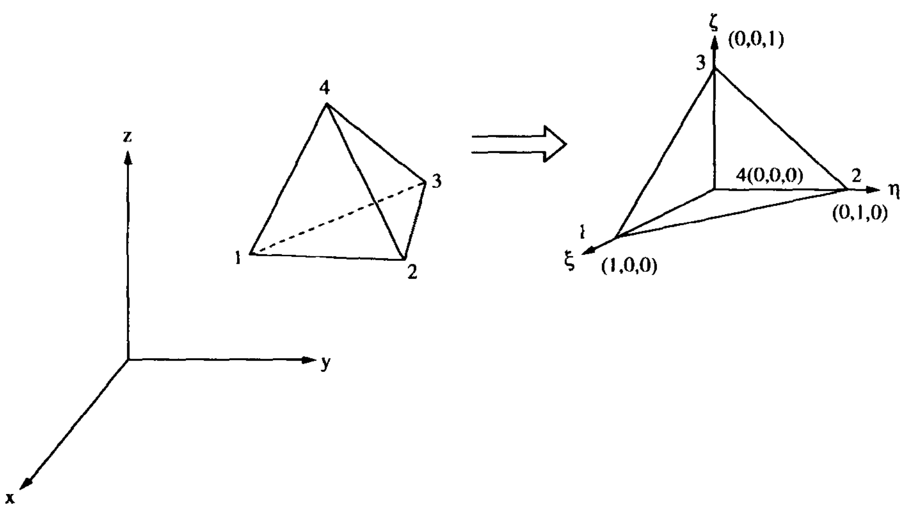

Fig. 1. Three-dimensional mapping of an arbitrary tetrahedron in X, Y, Z space into an unit orthogonal tetrahedron in ξ, η, ζ, space.

Let us now rewrite eqn (12) in an alternative form so that

$$III_V^{\alpha\beta\gamma} = |\det J| \iiint_V f(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \tag{13}$$

where

$$f(\xi, \eta, \zeta) = X(\xi, \eta, \zeta) Y(\xi, \eta, \zeta) Z(\xi, \eta, \zeta) \tag{14a}$$

with

$$X(\xi, \eta, \zeta) = x^{\alpha}(\xi, \eta, \zeta), \quad Y(\xi, \eta, \zeta) = y^{\beta}(\xi, \eta, \zeta)$$

$$Z(\xi, \eta, \zeta) = z^{\gamma}(\xi, \eta, \zeta). \tag{14b}$$

Now using the multinominal theorem and eqns (9) and (14a), (14b) it can be shown that

$$X(\xi, \eta, \zeta) = \sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4=\alpha} \frac{\lfloor\underline{\alpha}}{\lfloor\underline{\alpha_1}\,\lfloor\underline{\alpha_2}\,\lfloor\underline{\alpha_3}\,\lfloor\underline{\alpha_4}} \times x_4^{\alpha_1} x_{14}^{\alpha_2} x_{24}^{\alpha_3} x_{34}^{\alpha_4} \xi^{\alpha_2} \eta^{\alpha_3} \zeta^{\alpha_4} \tag{15}$$

$$Y(\xi, \eta, \zeta) = \sum_{\beta_1+\beta_2+\beta_3+\beta_4=\beta} \frac{\lfloor\underline{\beta}}{\lfloor\underline{\beta_1}\,\lfloor\underline{\beta_2}\,\lfloor\underline{\beta_3}\,\lfloor\underline{\beta_4}} \times y_4^{\beta_1} y_{14}^{\beta_2} y_{24}^{\beta_3} y_{34}^{\beta_4} \xi^{\beta_2} \eta^{\beta_3} \zeta^{\beta_4} \tag{16}$$

$$Z(\xi, \eta, \zeta) = \sum_{\gamma_1+\gamma_2+\gamma_3+\gamma_4=\gamma} \frac{\lfloor\underline{\gamma}}{\lfloor\underline{\gamma_1}\,\lfloor\underline{\gamma_2}\,\lfloor\underline{\gamma_3}\,\lfloor\underline{\gamma_4}} \times z_4^{\gamma_1} z_{14}^{\gamma_2} z_{24}^{\gamma_3} z_{34}^{\gamma_4} \xi^{\gamma_2} \eta^{\gamma_3} \zeta^{\gamma_4}. \tag{17}$$

We can now make use of the well known Taylor's theorem to expand the function $f(\xi, \eta, \zeta)$ in powers of ξ η, ζ and then we obtain

$$f(\xi, \eta, \zeta) = f(0, 0, 0) + \sum_{n=1}^{\alpha+\beta+\gamma} \frac{1}{\lfloor\underline{n}} \times \left[\left(\xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + \zeta \frac{\partial}{\partial \zeta} \right)^n f(\xi, \eta, \zeta) \right]. \tag{18}$$

Let us now again use the multinominal theorem in eqn (18) to obtain

$$f(\xi, \eta, \zeta) = f(0, 0, 0) + \sum_{n=1}^{\alpha+\beta+\zeta} \sum_{n_1+n_2+n_3=n} \times \left[\frac{\partial^{n_1+n_2+n_3} f(\xi, \eta, \zeta)}{\partial \xi^{n_1}\, \partial \eta^{n_2}\, \partial \zeta^{n_3}} \Big/ (\lfloor\underline{n_1}\,\lfloor\underline{n_2}\,\lfloor\underline{n_3}) \right]_{(0,0,0)} \times \xi^{n_1} \eta^{n_2} \zeta^{n_3}. \tag{19}$$

We shall now use generalized form of Leibnitz's theorem (see Appendix) to obtain the following:

$$\frac{\partial^{n_1}}{\partial \xi^{n_1}} (f(\xi, \eta, \zeta)) = \sum_{k_1+k_2+k_3=n_1} \frac{\lfloor\underline{n_1}}{\lfloor\underline{k_1}\,\lfloor\underline{k_2}\,\lfloor\underline{k_3}} \times \left(\frac{\partial^{k_1} X}{\partial \xi^{k_1}} \right) \left(\frac{\partial^{k_2} Y}{\partial \xi^{k_2}} \right) \left(\frac{\partial^{k_3} Z}{\partial \xi^{k_3}} \right) \tag{20a}$$

$$\frac{\partial^{n_1+n_2}}{\partial \xi^{n_1}\, \partial \eta^{n_2}} (f(\xi, \eta, \zeta)) = \sum_{k_1+k_2+k_3=n_1} \frac{\lfloor\underline{n_1}}{\lfloor\underline{k_1}\,\lfloor\underline{k_2}\,\lfloor\underline{k_3}} \sum_{l_1+l_2+l_3=n_2} \frac{\lfloor\underline{n_2}}{\lfloor\underline{l_1}\,\lfloor\underline{l_1}\,\lfloor\underline{l_1}} \times \left(\frac{\partial^{k_1+l_1} X}{\partial \xi^{k_1}\, \partial \eta^{l_1}} \right) \left(\frac{\partial^{k_2+l_2} Y}{\partial \xi^{k_2}\, \partial \eta^{l_2}} \right) \left(\frac{\partial^{k_3+l_3} Z}{\partial \xi^{k_3}\, \partial \eta^{l_3}} \right) \tag{20b}$$

$$\frac{\partial^{n_1+n_2+n_3}}{\partial\xi^{n_1}\,\partial\eta^{n_2}\,\partial\zeta^{n_3}}(f(\xi,\eta,\zeta))$$
$$= \sum_{k_1+k_2+k_3=n_1}\frac{\lfloor n_1}{\lfloor k_1\,\lfloor k_2\,\lfloor k_3}\left(\sum_{l_1+l_2+l_3=n_2}\frac{\lfloor n_2}{\lfloor l_1\,\lfloor l_2\,\lfloor l_3}\right)$$
$$\times\left(\sum_{m_1+m_2+m_3=n_3}\frac{\lfloor n_3}{\lfloor m_1\,\lfloor m_2\,\lfloor m_3}\right)\left(\frac{\partial^{k_1+l_1+m_1}_{X}}{\partial\xi^{k_1}\,\partial\eta^{l_1}\,\partial\zeta^{m_1}}\right)$$
$$\times\left(\frac{\partial^{k_2+l_2+m_2}_{Y}}{\partial\xi^{k_2}\,\partial\eta^{l_2}\,\partial\zeta^{m_2}}\right)\left(\frac{\partial^{k_3+l_3+m_2}_{Z}}{\partial\xi^{k_3}\,\partial\eta^{l_3}\,\partial\zeta^{m_3}}\right). \tag{20c}$$

From eqns (5), (15), (16), (17) and (20c), we obtain

$$\left(\frac{\partial^{n_1+n_2+n_3}f(\xi,\eta,\zeta)}{\partial\xi^{n_1}\,\partial\eta^{n_2}\,\partial\zeta^{n_3}}\Big/(\lfloor n_1\,\lfloor n_2\,\lfloor n_3)\right)_{(0,0,0)}$$
$$= G(n_1,n_2,n_3)$$
$$= \sum_{k_1+k_2+k_3=n_1}\frac{1}{\lfloor k_1\,\lfloor k_2\,\lfloor k_3}$$
$$\times\sum_{l_1+l_2+l_3=n_2}\frac{1}{\lfloor l_1\,\lfloor l_2\,\lfloor l_3}$$
$$\times\sum_{m_1+m_2+m_3=n_3}\frac{1}{\lfloor m_1\,\lfloor m_2\,\lfloor m_3}$$
$$\times\lambda_{k_1l_1m_1}\,\mu_{k_2l_2m_2}\,\delta_{k_3l_3m_3} \tag{21}$$

where

$$\lambda_{k_1l_1m_1} = \left(\frac{\partial^{k_1+l_1+m_1}X}{\partial\xi^{k_1}\,\partial\eta^{l_1}\,\partial\zeta^{m_1}}\right)_{(0,0,0)}$$
$$= \frac{\lfloor\alpha}{\lfloor(\alpha-k_1-l_1-m_1)}$$
$$\times x_4^{\alpha-k_1-l_1-m_1}x_{14}^{k_1}x_{24}^{l_1}x_{34}^{m_1},$$
$$(k_1+l_1+m_1\leqslant\alpha) \tag{22a}$$

$$\mu_{k_2l_2m_2} = \left(\frac{\partial^{k_2+l_2+m_2}Y}{\partial\xi^{k_2}\,\partial\eta^{l_2}\,\partial\zeta^{m_2}}\right)_{(0,0,0)} = \frac{\lfloor\beta}{\lfloor\beta-k_2-l_2-m_2}$$
$$\times y_4^{\beta-k_2-l_2-m_2}y_{14}^{k_2}y_{24}^{l_2}y_{34}^{m_2},$$
$$(k_2+l_2+m_2\leqslant\beta) \tag{22b}$$

$$\delta_{k_3l_3m_3} = \left(\frac{\partial^{k_3+l_3+m_3}Z}{\partial\xi^{k_3}\,\partial\eta^{l_3}\,\partial\zeta^{m_3}}\right) = \frac{\lfloor\gamma_1}{\lfloor\gamma-k_3-l_3-m_3}$$
$$\times z_4^{\gamma-k_3-l_3-m_3}z_{14}^{k_3}z_{24}^{l_3}z_{34}^{m_3},$$
$$(k_3+l_3+m_3\leqslant\gamma). \tag{22c}$$

Using eqns (9), (14a, b), we also obtain

$$f(0,0,0) = x_4^{\alpha}y_4^{\beta}z_4^{\gamma}. \tag{23}$$

Now substituting from eqns (21) and (23) into eqn (19), we obtain

$$f(\xi,\eta,\zeta) = x_4^{\alpha}y_4^{\beta}z_4^{\gamma} + \sum_{n=1}^{\alpha_1+\beta+\gamma}\sum_{n_1+n_2+n_3=n}$$
$$\times G(n_1,n_2,n_3)\xi^{n_1}\eta^{n_2}\zeta^{n_3}. \tag{24}$$

Substituting the result of eqn (24) into eqn (13) and performing integration, we obtain the result of eqn (2). This completes the proof of theorem 1.

Example

We consider as an example, the integral

$$III_V^{2,1,0} = \iiint_V x^2y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{25}$$

where V is the tetrahedron in R^3 with vertices

$$\langle(5,5,0),(10,10,0),(8,7,8),(10,5,0)\rangle.$$

The algorithm stated in eqn (2) completes the integral of eqn (25) as

$$III_V^{2,1,0} = |\det J|\{\tfrac{1}{6}x_4^2y_4 + G(1,0,0)III_V^{1,0,0}$$
$$+G(0,1,0)III_V^{0,1,0} + G(0,0,1)III_V^{0,0,1}$$
$$+G(1,1,0)III_V^{1,1,0} + G(1,0,1)III_V^{1,0,1}$$
$$+G(0,1,1)III_V^{0,1,1} + G(2,0,0)III_V^{2,0,0}$$
$$+G(0,2,0)III_V^{0,2,0} + G(0,0,2)III_V^{0,0,2}$$
$$+G(1,1,1)III_V^{1,1,1} + G(2,1,0)III_V^{2,1,0}$$
$$+G(0,2,1)III_V^{0,2,1} + G(1,0,2)III_V^{1,0,2}$$
$$+G(2,0,1)III_V^{2,0,1} + G(0,1,2)III_V^{0,1,2}$$
$$+G(1,2,0)III_V^{1,2,0} + G(3,0,0)III_V^{3,0,0}$$
$$+G(0,3,0)III_V^{0,3,0} + G(0,0,3)III_V^{0,0,3}\}. \tag{26}$$

From eqn (11), we obtain

$$|\det J| = 200. \tag{27}$$

From eqn (21), we obtain

$$G(n_1,n_2,n_3) = \sum_{k_1+k_2=n_1}\frac{1}{\lfloor k_1\,\lfloor k_2}\sum_{l_1+l_2=n_2}\frac{1}{\lfloor l_1\,\lfloor l_2}$$
$$\times\sum_{m_1+m_2=n_3}\frac{1}{\lfloor m_1\,\lfloor m_2}\lambda_{k_1l_1m_1}\mu_{k_2l_2m_2}. \tag{28}$$

From eqns (28), we can obtain

$$G(1,0,0)=\lambda_{1,0,0}\mu_{0,0,0}+\lambda_{0,0,0}\mu_{1,0,0}$$
$$G(0,1,0)=\lambda_{0,1,0}\mu_{0,0,0}+\lambda_{0,0,0}\mu_{0,1,0}$$
$$G(0,0,1)=\lambda_{0,0,1}\mu_{0,0,0}+\lambda_{0,0,0}\mu_{0,0,1}$$
$$G(1,1,0)=\lambda_{1,1,0}\mu_{0,0,0}+\lambda_{1,0,0}\mu_{0,1,0}+\lambda_{0,1,0}\mu_{1,0,0}$$
$$G(1,0,1)=\lambda_{1,0,1}\mu_{0,0,0}+\lambda_{1,0,0}\mu_{0,0,1}+\lambda_{0,0,1}\mu_{1,0,0}$$
$$G(0,1,1)=\lambda_{0,1,1}\mu_{0,0,0}+\lambda_{0,1,0}\mu_{0,0,1}+\lambda_{0,0,1}\mu_{0,1,0}$$
$$G(2,0,0)=\lambda_{1,0,0}\mu_{1,0,0}+\tfrac{1}{2}\lambda_{2,0,0}\mu_{0,0,0}$$
$$G(0,2,0)=\lambda_{0,1,0}\mu_{0,1,0}+\tfrac{1}{2}\lambda_{0,2,0}\mu_{0,0,0}$$
$$G(0,0,2)=\lambda_{0,0,1}\mu_{0,0,1}+\tfrac{1}{2}\lambda_{0,0,2}\mu_{0,0,0}$$
$$G(1,1,1)=\lambda_{1,1,0}\mu_{0,0,1}+\lambda_{1,0,1}\mu_{0,1,0}+\lambda_{0,1,1}\mu_{1,0,0}$$
$$G(2,1,0)=\tfrac{1}{2}\lambda_{2,0,0}\mu_{0,1,0}+\lambda_{1,1,0}\mu_{1,0,0}$$
$$G(0,2,1)=\tfrac{1}{2}\lambda_{0,2,0}\mu_{0,0,1}+\lambda_{0,1,1}\mu_{0,1,0}$$
$$G(1,0,2)=\lambda_{1,0,1}\mu_{0,0,1}+\tfrac{1}{2}\lambda_{0,0,2}\mu_{1,0,0}$$
$$G(2,0,1)=\tfrac{1}{2}\lambda_{2,0,0}\mu_{0,0,1}+\lambda_{1,0,1}\mu_{1,0,0}$$
$$G(0,1,2)=\lambda_{0,1,1}\mu_{0,0,1}+\tfrac{1}{2}\lambda_{0,0,2}\mu_{0,1,0}$$
$$G(1,2,0)=\lambda_{1,1,0}\mu_{0,1,0}+\tfrac{1}{2}\lambda_{0,2,0}\mu_{1,0,0}$$
$$G(3,0,0)=\tfrac{1}{2}\lambda_{2,0,0}\mu_{1,0,0}$$
$$G(0,3,0)=\tfrac{1}{2}\lambda_{0,2,0}\mu_{0,1,0}$$
$$G(0,0,3)=\tfrac{1}{2}\lambda_{0,0,2}\mu_{0,0,1}\,. \tag{29}$$

From eqn (22a, b), we can obtain

$$\lambda_{0,0,0}=x_4^2=100,\quad \lambda_{1,0,0}=2x_4x_{14}=-100,$$
$$\lambda_{0,1,0}=2x_4x_{24}=0,\quad \lambda_{0,0,1}=2x_4x_{34}=-40,$$
$$\lambda_{1,1,0}=2x_{14}x_{24}=0,\quad \lambda_{1,0,1}=2x_{14}x_{34}=20,$$
$$\lambda_{0,1,1}=2x_{24}x_{34}=0,\quad \lambda_{2,0,0}=2x_{14}^2=50,$$
$$\lambda_{0,2,0}=2x_{24}^2=0,\qquad \lambda_{0,0,2}=2x_{34}^2=8,$$
$$\mu_{0,0,0}=y_4=5,\quad \mu_{1,0,0}=y_{14}=0,$$
$$\mu_{0,1,0}=y_{24}=5,\quad \mu_{0,0,1}=y_{34}=2. \tag{30}$$

We also recall from eqn (7) that

$$III_V^{n_1,n_2,n_3}=\frac{\underline{/n_1}\,\underline{/n_2}\,\underline{/n_3}}{(n_1+n_2+n_3+3)!}\,. \tag{31}$$

Substituting the numerical values from eqns (27)–(31) into eqn (26), we obtain:

$$\begin{aligned}III_V^{2,1,0}&=200\{500\times\tfrac{1}{6}+(-500)\times\tfrac{1}{24}\\&+500\times\tfrac{1}{24}+O\times\tfrac{1}{24}+(-500)\tfrac{1}{120}+(-100)\\&\times\tfrac{1}{120}+(-200)\times\tfrac{1}{120}+125\times\tfrac{1}{60}\\&+0\times\tfrac{1}{60}+(-60)+100\times\tfrac{1}{720}+125\\&\times\tfrac{1}{360}+0\tfrac{1}{360}+40\times\tfrac{1}{360}+50\\&\times\tfrac{1}{360}+20\times\tfrac{1}{360}+0\times\tfrac{1}{360}+0\times\tfrac{1}{120}+0\\&\times\tfrac{1}{120}+8\times\tfrac{1}{120}\}=\frac{47165}{3}.\end{aligned} \tag{32}$$

The result obtained in eqn (32) is in agreement with Bernardini [6]. We see that the present algorithm saves about 60% arithmetic operations, as compared to the previous work.

SURFACE INTEGRATION OVER AN ARBITRARY TETRAHEDRON

The volume integral of a scalar function $f(P)=x^\alpha y^\beta z^\gamma$ (α,β,γ positive integers) can be easily derived by using the Gauss's divergence theorem, as shown by the following theorem.

Theorem 2

Let V be a three-dimensional tetrahedron bounded by a tetrahedral surface S. Then the structure product $III_V^{\alpha\beta\gamma}$ over a linear three-tetrahedron (linear arbitrary tetrahedron in three-dimensional space) is given by the equation:

$$\begin{aligned}III_V^{\alpha\beta\gamma}&=\iiint_V x^\alpha y^\beta z^\gamma\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\&=\frac{|\det J|}{(\alpha+1)\det J}\iint_{\bar\tau}\{A(u,\vartheta)-B(u,\vartheta)\\&\quad-C(u,\vartheta)-D(u,\vartheta)\}\,\mathrm{d}u\,\mathrm{d}\vartheta,\end{aligned} \tag{33a}$$

where $\bar\tau$ is the unit triangle $\langle(0,0),(1,0),(0,1)\rangle$ in the $u\vartheta$-plane and $A(u,\vartheta)$, $B(u,\vartheta)$, $C(u,\vartheta)$ and $D(u,\vartheta)$ are explained in the body of the following proof of this theorem.

Proof: we have from eqns (8)–(12)

$$III_V^{\alpha\beta\gamma}=|\det J|\iiint_V x^\alpha(\xi,\eta,\zeta)\times y^\beta(\xi,\eta,\zeta)z^\gamma(\xi,\eta,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{33b}$$

where $\bar{V}$ is the unit orthogonal tetrahedron

$$\langle(0,0,0),(1,0,0),(0,1,0),(0,0,1)\rangle.$$

We can also write eqn (33b) in an alternative form as

$$III_{\bar{V}}^{\alpha\beta\gamma} = \frac{|\det J|}{(\alpha+1)} \iiint_{\bar{V}} \frac{\partial}{\partial x}\{x^{\alpha+1}(\xi,\eta,\zeta) \times y^{\beta}(\xi,\eta,\zeta) z^{\gamma}(\xi,\eta,\zeta)\}\, d\xi\, d\eta\, d\zeta. \quad (34)$$

Using the chain rule on partial differentiation we can rewrite eqn (34) as

$$III_{\bar{V}}^{\alpha\beta\gamma} = \frac{|\det J|}{(\alpha+1)(\det J)} \iiint_{\bar{V}} \left\{\frac{\partial}{\partial \xi}\left(x^{\alpha+1}y^{\beta}z^{\gamma}\frac{\partial(y,z)}{\partial(\eta,\zeta)}\right) + \frac{\partial}{\partial \eta}\left(-x^{\alpha+1}y^{\beta}z^{\gamma}\frac{\partial(y,z)}{\partial(\xi,\zeta)}\right) + \frac{\partial}{\partial z} \times \left(x^{\alpha+1}y^{\beta}z^{\gamma}\frac{\partial(y,z)}{\partial(\xi,\eta)}\right)\right\} d\xi\, d\eta\, d\zeta$$

$$= \frac{1}{(\alpha+1)}\frac{|\det J|}{(\det J)} \iint_{\bar{S}} F\cdot \mathbf{n}\, d\bar{S} \quad (35)$$

where $\bar{S}$ is the surface of the unit orthogonal tetrahedron $\langle(0,0,0),(1,0,0),(0,1,0),(0,0,1)\rangle$ and $\mathbf{n}$ is the unit normal vector pointing outward to $\bar{V}$.

$$F_1 = x^{\alpha+1}y^{\beta}z^{\gamma}\frac{\partial(y,z)}{\partial(\eta,\zeta)}, \quad F_2 = -x^{\alpha+1}y^{\beta}z^{\gamma}\frac{(y,z)}{\partial(\xi,\zeta)}$$

$$F_3 = x^{\alpha+1}y^{\beta}z^{\gamma}\frac{\partial(y,z)}{\partial(\xi,\eta)}. \quad (36)$$

Clearly from eqn (9), we can find

$$\frac{\partial(y,z)}{\partial(\eta,\zeta)} = \begin{vmatrix} y_{24} & y_{34} \\ z_{24} & z_{34} \end{vmatrix},$$

$$-\frac{\partial(y,z)}{\partial(\xi,\zeta)} = \begin{vmatrix} y_{34} & y_{14} \\ z_{34} & z_{14} \end{vmatrix} = \frac{\partial(y,z)}{\partial(\zeta,\xi)},$$

$$\frac{\partial(y,z)}{\partial(\xi,\eta)} = \begin{vmatrix} \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{vmatrix} = \begin{vmatrix} y_{14} & y_{24} \\ z_{14} & z_{24} \end{vmatrix}. \quad (37)$$

In order to obtain a working relationship of eqn (35), let us examine the surface integral

$$\iint_{\bar{S}} \mathbf{F}\cdot\mathbf{n}\, d\bar{S},$$

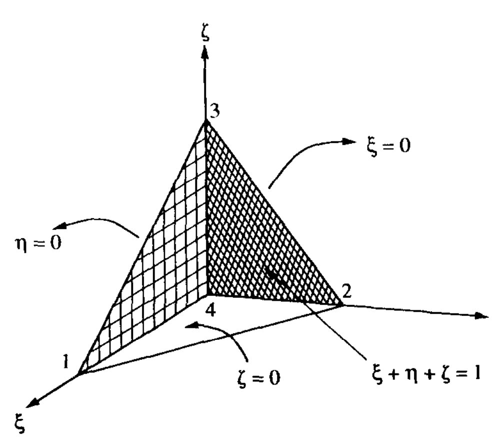

Fig. 2. The unit orthogonal tetrahedron in ξ, η, ζ, space.

now clearly from Fig. 2 $\bar{S}$ consists of four triangular surfaces:

$$\bar{S}_1 = \Delta_{123}, \quad \bar{S}_2 = \Delta_{423}, \quad \bar{S}_3 = \Delta_{413} \quad \text{and } \bar{S}_4 = \Delta_{412}$$

where Δ_{ijk} means the triangular surface formed by vertices i, j, k. Thus we can write

$$\iint_{\bar{S}} \mathbf{F}\cdot\mathbf{n}\, d\bar{s} = \sum_{i=1}^{4} \iint_{\bar{S}_i} \mathbf{F}\cdot\mathbf{n}_i\, d\bar{S}_i \quad (38)$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ are outward pointing unit normal vectors to $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$ and $\bar{S}_4$, respectively. By considering the projection of $\bar{S}_1$ on $\xi\eta$ plane, and the equation of surface

$$\bar{S}_1 : \xi + \eta + \zeta - 1 = 0,$$

we find:

$$\iint_{\bar{S}_i} F\cdot\mathbf{n}_1\, d\bar{S}_1 = \int_0^1 \int_0^{1-\xi} \sum_{i=1}^{3} F_i(\xi,\eta,1-\xi-\eta)\, d\xi\, d\eta. \quad (39)$$

Similarly, we can show that

$$\iint_{\bar{S}_2} \mathbf{F}\cdot\mathbf{n}_2\, d\bar{S}_2 = -\int_0^1 \int_0^{1-\eta} F_1(0,\eta,\zeta)\, d\eta\, d\zeta$$

$$\iint_{\bar{S}_3} \mathbf{F}\cdot\mathbf{n}_3\, d\bar{S}_3 = -\int_0^1 \int_0^{1-\xi} F_2(\xi,0,\zeta)\, d\xi\, d\zeta$$

$$\iint_{\bar{S}_4} \mathbf{F}\cdot\mathbf{n}_4\, d\bar{S}_4 = -\int_0^1 \int_0^{1-\xi} F_3(\xi,\eta,0)\, d\xi\, d\eta. \quad (40)$$

Substituting eqns (39) and (40) into eqn (38) we obtain

$$\iint_{\bar{S}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}\bar{S} = \int_0^1\int_0^{1-\xi}\sum_{i=1}^{3} F_i(\xi,\eta,1-\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$-\int_0^1\int_0^{1-\eta} F_1(0,\eta,\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

$$-\int_0^1\int_0^{1-\xi} F_2(\xi,0,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta$$

$$-\int_0^1\int_0^{1-\xi} F_3(\xi,\eta,0)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{41}$$

From eqns (35), (36), (37) and (41) we can show that

$$\iiint_V x^\alpha y^\beta z^\gamma\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$=\frac{1}{(\alpha+1)}\frac{|\det J|}{(\det J)}\iint_{\bar{\tau}}\{A(u,\vartheta)-B(u,\vartheta)$$

$$-C(u,\vartheta)-D(u,\vartheta)\}\,\mathrm{d}u\,\mathrm{d}\vartheta \tag{42}$$

where $\bar{\tau}$ is the unit triangle $\langle(0,0),(1,0),(0,1)\rangle$ in the u, ϑ plane.

$$A(u,\vartheta)=\left\{\frac{\partial(y,z)}{\partial(\eta,\zeta)}-\frac{\partial(y,z)}{\partial(\xi,\zeta)}+\frac{\partial(y,z)}{\partial(\xi,\eta)}\right\}$$

$$\times\{x^{\alpha+1}(u,\vartheta,1-u-\vartheta)$$

$$\times y^\beta(u,\vartheta,1-u-\vartheta)z^\gamma(u,\vartheta,1-u,\vartheta)\}$$

$$B(u,\vartheta)=\frac{\partial(y,z)}{\partial(\eta,\zeta)}x^{\alpha+1}(0,u,\vartheta)y^\beta(0,u,\vartheta)z^\gamma(0,u,\vartheta),$$

$$C(u,\vartheta)=\frac{\partial(y,z)}{\partial(z,\xi)}x^{\alpha+1}(u,0,\vartheta)y^\beta(u,0,\vartheta)z^\gamma(u,0,\vartheta),$$

$$D(u,\vartheta)=\frac{\partial(y,z)}{\partial(\xi,\eta)}x^{\alpha+1}(u,\vartheta,0)y^\beta(u,\vartheta,0)z^\gamma(u,\vartheta,0). \tag{43}$$

$$\left.\begin{aligned} x(u,\vartheta,1-u-\vartheta)&=x_3+x_{13}u+x_{23}\vartheta\\ y(u,\vartheta,1-u-\vartheta)&=y_3+y_{13}u+y_{23}\vartheta\\ z(u,\vartheta,1-u-\vartheta)&=z_3+z_{13}u+z_{23}\vartheta\end{aligned}\right\} \tag{44a}$$

$$\left.\begin{aligned} x(0,u,\vartheta)&=x_4+x_{24}u+x_{34}\vartheta\\ y(0,u,\vartheta)&=y_4+y_{24}u+y_{34}\vartheta\\ z(0,u,\vartheta)&=z_4+z_{24}u+z_{34}\vartheta\end{aligned}\right\} \tag{44b}$$

$$\left.\begin{aligned} x(u,0,\vartheta)&=x_4+x_{14}u+x_{34}\vartheta\\ y(u,0,\vartheta)&=y_4+y_{14}u+y_{34}\vartheta\\ z(u,0,\vartheta)&=z_4+z_{14}u+z_{34}\vartheta\end{aligned}\right\} \tag{44c}$$

$$\left.\begin{aligned} x(u,\vartheta,0)&=x_4+x_{14}u+x_{24}\vartheta\\ y(u,\vartheta,0)&=y_4+y_{14}u+y_{24}\vartheta\\ z(u,\vartheta,0)&=z_4+z_{14}u+z_{24}\vartheta\end{aligned}\right\} \tag{44d}$$

This completes the proof of theorem 2.

The evaluation of integrals in eqns (42)–(44)(a–d) can be computed if we derive a working relationship for the integral.

$$II_{\bar{\tau}}^{\alpha\beta\gamma}=\iint_{\bar{\tau}} x^\alpha(u,\vartheta)y^\beta(u,\vartheta)z^\gamma(u,\vartheta)\,\mathrm{d}u\,\mathrm{d}\vartheta, \tag{45}$$

where

$$x(u,\vartheta)=x_0+(x_a-x_0)u+(x_b-x_0)\vartheta$$

$$y(u,\vartheta)=y_0+(y_a-y_0)u+(y_b-y_0)\vartheta$$

$$z(u,\vartheta)=z_0+(z_a-z_0)u+(z_b-z_0)\vartheta. \tag{46}$$

This working formulae for eqn (45) is contained in the following theorem.

Theorem 3

A structure product

$$II_{\bar{\tau}}^{\alpha\beta\gamma}$$

over a unit triangle $\bar{\tau}=\langle(0,0),(1,0),(0,1)\rangle$ is a polynomial combination of the coordinates (x_0,y_0,z_0), (x_a,y_a,z_a), (x_b,y_b,z_b) and is given by

$$II_{\bar{\tau}}^{\alpha\beta\gamma}=\frac{x_0^\alpha y_0^\beta z_0^\gamma}{2}+\sum_{k=1}^{\alpha+\beta+\gamma}\sum_{\vartheta+\zeta=k}\lambda_{\vartheta s}II^{\vartheta S} \tag{47}$$

$$II^{\vartheta,s}=\iint_{\bar{\tau}} u^\vartheta\theta^s\,\mathrm{d}u\,\mathrm{d}\vartheta=\frac{\underline{/\vartheta}\ \underline{/s}}{\underline{/(\vartheta+s+2)}} \tag{48}$$

$$\lambda_{\vartheta s}=\left\{\frac{\partial^{\vartheta+s}[x^\alpha(u,\vartheta)y^\beta(u,\vartheta)z^\gamma(u,\vartheta)]}{\partial u^\vartheta\,\partial\vartheta^s}\Big/(\underline{/\vartheta}\ \underline{/s}\right\}_{(0,0)}$$

$$=\sum_{\alpha_1+\alpha_2+\alpha_3=\vartheta},\ \sum_{\beta_1+\beta_2+\beta_3=s} a_{\alpha_1,\beta_1}^{\alpha-\alpha_1-\beta_1}$$

$$\times b_{\alpha_2\beta_2}^{\beta-\alpha_2-\beta_2}c_{\alpha_3,\beta_3}^{\gamma-\alpha_3-\beta_3}, \tag{49}$$

with

$$a_{l_2,l_3}^{l_1}=\frac{\underline{/\alpha}}{\underline{/l_1}\,\underline{/l_2}\,\underline{/l_3}}x_0^{l_1}a_x^{l_2}b_x^{l_3},\quad l_1+l_2+l_3=\alpha$$

$$b_{m_2,m_3}^{m_1}=\frac{\underline{/\beta_1}}{\underline{/m_1}\,\underline{/m_2}\,\underline{/m_3}}y_0^{m_1}a_y^{m_2}b_y^{m_2},\quad m_1+m_2+m_3=\beta$$

$$c_{n_2,n_3}^{n_1}=\frac{\underline{/\gamma}}{\underline{/n_1}\,\underline{/n_2}\,\underline{/n_3}}z_s^{n_1}a_z^{n_2}b_z^{n_3},\quad n_1+n_2+n_3=\gamma$$

$$a_x = x_a - x_0, \quad b_x = x_b - x_0,$$

$$a_y = y_a - y_0, \quad b_y = y_b - y_0,$$

$$a_z = z_a - z_0, \quad b_z = z_b - z_0. \tag{50}$$

Let us rewrite eqn (45) as

$$II_{\bar{\tau}}^{\alpha\beta\gamma} = \iint_{\bar{\tau}} f(u, \vartheta)\, \mathrm{d}u\, \mathrm{d}\vartheta \tag{51}$$

where

$$f(u, \vartheta) = X(u, \vartheta)Y(u, \vartheta)Z(u, \vartheta)$$

with

$$X(u, \vartheta) = x^{\alpha}(u, \vartheta) = (x_0 + a_x u + b_x \vartheta)^{\alpha}$$

$$Y(u, \vartheta) = y^{\beta}(u, \vartheta) = (y_0 + a_y u + b_y \vartheta)^{\beta}$$

$$Z(u, \vartheta) = z^{\gamma}(u, \vartheta) = (z_0 + a_z u + b_z \vartheta)^{\gamma} \tag{52}$$

using the multinominal theorem, we can rewrite eqn (50) as

$$X(u, \vartheta) = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} a_{\alpha_2, \beta_3}^{\alpha_1} u^{\alpha_2} \vartheta^{\alpha_3},$$

$$Y(u, \vartheta) = \sum_{\beta_1 + \beta_2 + \beta_3 = \beta} b_{\beta_2, \beta_3}^{\beta_1} u^{\beta_2} \vartheta^{\beta_3},$$

$$Z(u, \vartheta) = \sum_{\gamma_1 + \gamma_2 + \gamma_3 = \gamma} c_{\gamma_2 \gamma_3}^{\gamma_1} u^{\gamma_2} \vartheta^{\gamma_3} \tag{53}$$

and

$$a_{\alpha_2, \alpha_3}^{\alpha_1} = \frac{\underline{|\alpha}}{\underline{|\alpha_1}\, \underline{|\alpha_2}\, \underline{|\alpha_3}} x_0^{\alpha_1} a_x^{\alpha_2} b_x^{\alpha_3},$$

$$b_{\beta_2, \beta_3}^{\beta_1} = \frac{\underline{|\beta}}{\underline{|\beta_1}\, \underline{|\beta_2}\, \underline{|\beta_3}} y_0^{\beta_1} a_y^{\beta_2} b_y^{\beta_3}$$

$$c_{\gamma_2, \gamma_3}^{\gamma_1} = \frac{\underline{|\gamma}}{\underline{|\gamma_1}\, \underline{|\gamma_2}\, \underline{|\gamma_3}} z_0^{\gamma_1} a_z^{\gamma_2} b_z^{\gamma_3}. \tag{54}$$

We can now make use of the Taylor's theorem to expand the function about the point (0, 0), and then we obtain

$$f(u, \vartheta) = f(0, 0) + \sum_{k=1}^{\alpha+\beta+\gamma} \frac{1}{\underline{|k}} \left[\left(u \frac{\partial}{\partial u} + \vartheta \frac{\partial}{\partial \vartheta} \right)^k f(u, \vartheta) \right]_{(0,0)} \tag{55}$$

We again use the multinominal theorem in the above eqn (55) and obtain

$$f(u, \vartheta) = f(0, 0) + \sum_{k=1}^{\alpha+\beta+\gamma} \frac{1}{\underline{|k}} \sum_{\vartheta + s = k} \frac{\underline{|k}}{\underline{|\vartheta}\, \underline{|s}} \times \left(\frac{\partial^{\vartheta + s} f(u, \vartheta)}{\partial u^{\vartheta} \partial \vartheta^{s}} \right)_{(0,0)} u^{\vartheta} \vartheta^{s}. \tag{56}$$

We shall now use the generalized form of Leibnitz's theorem (see Appendix) to obtain the following:

$$\frac{\partial^{v}}{\partial u^{v}} \{f(u, \vartheta)\} = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \vartheta} \frac{\underline{|\vartheta}}{\underline{|\alpha_1}\, \underline{|\alpha_2}\, \underline{|\alpha_3}} \times \left(\frac{\partial^{\alpha_1} X}{\partial u^{\alpha_1}} \right) \left(\frac{\partial^{\alpha_2} Y}{\partial u^{\alpha_2}} \right) \left(\frac{\partial^{\alpha_3} zZ}{\partial u^{\alpha_3}} \right) \tag{57}$$

and then,

$$\frac{\partial^{s}}{\partial \vartheta^{s}} \left(\frac{\partial^{v} f(u, \vartheta)}{\partial u^{v}} \right) = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \vartheta} \frac{\underline{|\vartheta}}{\underline{|\alpha_1}\, \underline{|\alpha_2}\, \underline{|\alpha_3}} \frac{\partial^{s}}{\partial \vartheta^{s}} \times \left\{ \left(\frac{\partial^{\alpha_1} X}{\partial u^{\alpha_1}} \right) \left(\frac{\partial^{\alpha_2} Y}{\partial u^{\alpha_2}} \right) \left(\frac{\partial^{\alpha_3} Z}{\partial u^{\alpha_3}} \right) \right.$$

$$= \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \vartheta} \frac{\underline{|v}}{\underline{|\alpha_1}\, \underline{|\alpha_2}\, \underline{|\alpha_3}} \times \sum_{\beta_1 + \beta_2 + \beta_3 = s} \frac{\underline{|s}}{\underline{|\beta_1}\, \underline{|\beta_2}\, \underline{|\beta_3}} \times \left(\frac{\partial^{\alpha_1 + \beta_1} X}{\partial u^{\alpha_1} \partial \vartheta^{\beta_1}} \right) \left(\frac{\partial^{\alpha_2 + \beta_2} Y}{\partial u^{\alpha_2} \partial \vartheta^{\beta_2}} \right) \times \left(\frac{\partial^{\alpha_3 + \beta_3} Z}{\partial u^{\alpha_3} \partial \vartheta^{\beta_3}} \right). \tag{58}$$

From eqns (49) and (58), we obtain

$$\left\{ \frac{\partial^{\vartheta + s} f(u, \vartheta)}{\partial u^{\vartheta} \partial \vartheta^{s}} \Big/ (\underline{|\vartheta}\, \underline{|s}) \right\}_{(0,0)}$$

$$= \lambda_{\vartheta s} \text{ (definition)}$$

$$= \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \vartheta} \frac{1}{\underline{|\alpha_1}\, \underline{|\alpha_2}\, \underline{|\alpha_3}} \sum_{\beta_1 + \beta_2 + \beta_3 = s} \frac{1}{\underline{|\beta_1}\, \underline{|\beta_2}\, \underline{|\beta_3}} \times \left(\frac{\partial^{\alpha_1 + \beta_1} X}{\partial u^{\alpha_1} \partial \vartheta^{\beta_1}} \right)_{(0,0)} \left(\frac{\partial^{\alpha_2 + \beta_2} Y}{\partial u^{\alpha_2} \partial \vartheta^{\beta_2}} \right)_{(0,0)} \left(\frac{\partial^{\alpha_3 + \beta_3} Z}{\partial u^{\alpha_3} \partial \vartheta^{\beta_3}} \right)_{(0,0)} \tag{59}$$

Clearly from eqns (53) and (54), we obtain

$$\left(\frac{\partial^{\alpha_1 + \beta_1} X}{\partial u^{\alpha_1} \partial \vartheta^{\beta_1}} \right)_{(0,0)} = \underline{|\alpha_1}\, \underline{|\beta_1}\, a_{\alpha_1, \beta_1}^{\alpha - \alpha_1 - \beta_1}, \quad (\alpha_1 + \beta_1 \leqslant \alpha) \tag{60a}$$

$$\left(\frac{\partial^{\alpha_2 + \beta_2} Y}{\partial u^{\alpha_2} \partial \vartheta^{\beta_2}} \right)_{(0,0)} = \underline{|\alpha_2}\, \underline{|\beta_2}\, b_{\alpha_2, \beta_2}^{\beta - \alpha_2 - \beta_2}, \quad (\alpha_2 + \beta_2 \leqslant \beta) \tag{60b}$$

and

$$\left(\frac{\partial^{\alpha_3 + \beta_3} Z}{\partial u^{\alpha_3} \partial \vartheta^{\beta_3}} \right)_{(0,0)} = \underline{|\alpha_3}\, \underline{|\beta_3}\, c_{\alpha_3, \beta_3}^{\gamma - \alpha_3 - \beta_3}, \quad (\alpha_3 + \beta_3 \leqslant \gamma). \tag{60c}$$

Substituting from (61a–c) into eqn (59), we obtain

$$\lambda_{\vartheta s} = \left[\left\{ \frac{\partial^{\vartheta + s} f(u, \vartheta)}{\partial u^{v}\, \partial \vartheta^{s}} \right\} \Big/ \underline{|\vartheta}\, \underline{|s} \right]_{(0,0)}$$

$$= \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \vartheta} \; \sum_{\beta_1 + \beta_2 + \beta_3 = s} a_{\alpha_1, \beta_1}^{\alpha - \alpha_1 - \beta_1}$$

$$\times b_{\alpha_2, \beta_2}^{\beta - \alpha_2 - \beta_2} c_{\alpha_3, \beta_3}^{\gamma - \alpha_3 - \beta_3}. \qquad (61)$$

Thus, we can now rewrite eqn (56) as

$$f(u, \vartheta) = x_b^{\alpha} y_a^{\beta} z_0^{\gamma} + \sum_{k=1}^{\alpha + \beta + \gamma} \sum_{\vartheta + s = k} \lambda_{\vartheta s} u^{\vartheta} \vartheta^{s} \qquad (62)$$

where $\lambda_{\vartheta s}$ can be computed from eqn (61) now, finally from eqn (62), if we substitute the expression for $f(u, \vartheta)$ into eqn (51) and integrate over the unit triangle $\bar{\tau}$, we obtain the result stated in eqns (47) and (48). This completes the proof of theorem 3.

Example

We again consider as an example the integral of eqn (25). That is

$$III^{2,1,0} = \iiint_V x^2 y \, dx \, dy \, dz, \qquad (63)$$

where V is the tetrahedron in R^3 with vertices

$$\langle (5, 5, 0), (10, 10, 0), (8, 7, 8), (10, 5, 0) \rangle.$$

We shall now use the algorithm stated in eqn (42) to compute the above integral in eqn (63). Thus we have from eqns (4) and (37):

$$\det J = -200, \quad \frac{\partial(y, z)}{\partial(\eta\, \zeta)} = 40, \quad \frac{\partial(y, z)}{\partial(\xi, \zeta)} = 0.$$

$$\frac{\partial(y, z)}{\partial(\xi, \eta)} = 0. \qquad (64)$$

Hence from eqns (37) and (42), we obtain

$$III_V^{2,1,0} = -\frac{1}{3} \iint_{\bar{\tau}} \{A(u, \vartheta) - B(u, \vartheta)\} \, du \, d\vartheta, \qquad (65)$$

where

$$A(u, \vartheta) = 40(8 - 3u + 2\vartheta)^3 (7 - 2u + 3\vartheta) \qquad (66a)$$

$$B(u, \vartheta) = 40(10 - 2\vartheta)^3 (5 + 5u + 2\vartheta). \qquad (66b)$$

Let us now evaluate the following two integrals:

$$\iint_{\bar{\tau}} A(u, \vartheta) \, du \, d\vartheta = 40 \iint_{\bar{\tau}} (8 - 3u + 2\vartheta)^3 \times (7 - 2u + 3\vartheta) \, du \, d\vartheta$$

$$\iint_{\bar{\tau}} B(u, \vartheta) \, du \, d\vartheta = 40 \iint_{\bar{\tau}} (10 - 2\vartheta)^3 \times (5 + 5u + 2\vartheta) \, du \, d\vartheta \qquad (67a, b)$$

by the method of theorem 3.

From eqn (47), we should have for the evaluation of the above integrals $\alpha = 3$, $\beta = 1$ and hence, the computational scheme of theorem 3 is equivalent to the following:

$$\iint_{\bar{\tau}} x^3 y \, du \, d\vartheta = \frac{x_0^3 y_0}{2} + \tfrac{1}{6}\lambda_{1,0} + \tfrac{1}{6}\lambda_{0,1} + \tfrac{1}{12}\lambda_{2,0} + \tfrac{1}{12}\lambda_{0,2}$$
$$+ \tfrac{1}{24}\lambda_{1,1} + \tfrac{1}{20}\lambda_{3,0} + \tfrac{1}{20}\lambda_{0,3} + \tfrac{1}{60}\lambda_{2,1}$$
$$+ \tfrac{1}{60}\lambda_{1,2} + \tfrac{1}{30}\lambda_{4,0} + \tfrac{1}{30}\lambda_{0,4} + \tfrac{1}{120}\lambda_{3,1}$$
$$+ \tfrac{1}{120}\lambda_{1,3} + \tfrac{1}{180}\lambda_{2,2} \qquad (68)$$

where

$$\lambda_{1,0} = a_{1,0}^{2} b_{0,0}^{1} + a_{0,0}^{3} b_{1,0}^{1}$$

$$\lambda_{0,1} = a_{0,1}^{2} b_{0,0}^{1} + a_{0,0}^{3} b_{0,1}^{0},$$

Table 1. Table of numerical values required for computing the integrals

Expression/variable	$\iint_{\bar{\tau}} A(u, \vartheta) \, du \, d\vartheta$	$\iint_{\bar{\tau}} B(u, \vartheta) \, du \, d\vartheta$
x_0, a_x, b_x	8, −3, 2	10, 0, −2
y_0, a_y, b_y	7, −2, 3	5, 5, 2
$a_{0,0}^{3} = x_0^3$	$a_{0,0}^{3} = 512$	$a_{0,0}^{3} = 1000$
$a_{3,0}^{0} = a_x^3$	$a_{3,0}^{0} = -27$	$a_{30}^{0} = 0$
$a_{0,3}^{0} = b_x^3$	$a_{0,3}^{0} = 8$	$a_{0,3}^{0} = -8$
$a_{1,1}^{1} = 6x_0 a_x b_x$	$a_{1,1}^{1} = -288$	$a_{1,1}^{1} = 0$
$a_{1,0}^{2} = 3x_0^2 a_x$	$a_{1,0}^{2} = -576$	$a_{1,0}^{2} = 0$
$a_{2,0}^{1} = 3x_0 a_x^2$	$a_{2,0}^{1} = 216$	$a_{2,0}^{1} = 0$
$a_{0,1}^{2} = 3x_0^2 b_x$	$a_{0,1}^{2} = 384$	$a_{0,1}^{2} = -600$
$a_{0,2}^{1} = 3x_0 b_x^2$	$a_{0,2}^{1} = 96$	$a_{0,2}^{1} = 120$
$a_{2,1}^{0} = 3a_x^2 b_x$	$a_{2,1}^{0} = 54$	$a_{2,1}^{0} = 0$
$a_{1,2}^{0} = 3a_x b_1^2$	$a_{1,2}^{0} = -36$	$a_{1,2}^{0} = 0$

Now using the numerical values of Table 1, we can easily compute $\lambda_{\vartheta s}$ by using eqn (69).

Table 2. Table of numerical values $\lambda_{\vartheta s}$ required for computing the integrals

$\lambda_{\vartheta s}$	For integral $\iint_{\bar{\tau}} A(u, \vartheta)\, du\, d\vartheta$	For integral $\iint_{\bar{\tau}} B(u, \vartheta)\, du\, d\vartheta$
$\lambda_{1,0}$	−5056	5000
$\lambda_{0,1}$	4224	−1000
$\lambda_{2,0}$	2664	0
$\lambda_{0,2}$	1824	−600
$\lambda_{1,1}$	−4512	−3000
$\lambda_{3,0}$	−621	0
$\lambda_{0,3}$	344	200
$\lambda_{2,1}$	1602	0
$\lambda_{1,2}$	−1308	600
$\lambda_{4,0}$	54	0
$\lambda_{0,4}$	24	−16
$\lambda_{3,1}$	−189	0
$\lambda_{1,3}$	−124	−40
$\lambda_{2,2}$	234	0
$\frac{x_0^3 y_0}{2}$	1792	2500

$$\lambda_{2,0} = a^1_{2,0} b^1_{0,0} + a^2_{1,0} b^0_{1,0}$$

$$\lambda_{0,2} = a^1_{0,2} b^1_{0,0} + a^2_{0,1} b^0_{0,1}$$

$$\lambda_{1,1} = a^1_{1,1} b^1_{0,0} + a^2_{1,0} b^0_{0,1} + a^2_{0,1} b^1_{1,0}$$

$$\lambda_{3,0} = a^0_{3,0} b^1_{0,0} + a^1_{2,0} b^0_{1,0}$$

$$\lambda_{0,3} = a^0_{0,3} b^1_{0,0} + a^1_{0,2} b^0_{0,1}$$

$$\lambda_{2,1} = a^0_{2,1} b^1_{0,0} + a^1_{2,0} b^0_{1,1} + a^1_{1,1} b^0_{1,0}$$

$$\lambda_{1,2} = a^0_{1,2} b^1_{0,0} + a^1_{1,1} b^0_{0,1} + a^1_{0,2} b^0_{1,0}$$

$$\lambda_{0,4} = a^0_{0,3} b^0_{0,1}, \quad \lambda_{4,0} = a^0_{3,0} b^0_{1,0}$$

$$\lambda_{3,1} = a^0_{3,0} b^0_{0,1} + a^0_{2,1} b^0_{1,0}$$

$$\lambda_{1,3} = a^0_{1,2} b^0_{0,1} + a^0_{0,3} b^0_{1,0}$$

$$\lambda_{22} = a^0_{2,1} b^0_{0,1} + a^0_{1,2} b^0_{1,0}. \tag{69}$$

Using eqns (63), (67) and (68) and also the numerical values of $\lambda_{\vartheta s}$ tabulated in Table 2, we find

$$III_V^{2,1,0} = -\tfrac{40}{3}[1792 + \tfrac{1}{6}(-5056 + 4224) + \tfrac{1}{12}(2664 + 1824) + \tfrac{1}{24}(-4512) + \tfrac{1}{20}(-621 + 344) + \tfrac{1}{60}(1602 - 1308) + \tfrac{1}{30}(54 + 24) + \tfrac{1}{120}(-189 - 124) + \tfrac{1}{180}(234)] + [\tfrac{40}{3}\{2500 + \tfrac{1}{6}(5000 - 1000) + \tfrac{1}{12}(-600) + \tfrac{1}{24}(-3000) + \tfrac{1}{20}(200) + \tfrac{1}{60}(600) + \tfrac{1}{30}(-16) + \tfrac{1}{120}(-40)\}]$$

$$= \frac{47165}{3}. \tag{70}$$

The result obtained in eqn (70) is again the same as that in eqn (32). We wish to say that here again the present computational scheme is more efficient than the previous work [6].

CONCLUSIONS

The theorems we have presented on volume and surface integration are interesting for various reasons. Our formulas are more compact than the previous researchers and require less computer arithmetic, as is evident by comparing the summations required in earlier studies and the present one. We have developed a new technique to expand spatial expression $x^\alpha y^\beta z^\gamma$ in terms of two variables (for area integral) and three variables (for volume integral). This has clearly demonstrated the use of Taylor series expansion, a generalized form of Leibnitz's theorem and multinominal theorem. We have also included the proof of Leibnitz's theorem in its present form, which we feel is not available in standard text books. Explicit formulas for computing integrals of polynomials over an arbitrary tetrahedron are given. Two different approaches are discussed: one uses a direct mapping to transform the arbitrary tetrahedron into a unit orthogonal tetrahedron, while the other uses a boundary representation of the tetrahedron. These derivations are followed by a numerical example which explains the computational scheme, the accuracy and efficiency of the present integration formulas.

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APPENDIX

GENERALIZED FORM OF LEIBNITZ'S THEOREM (DIFFERENTIATION)

If $u_1(t), u_2(t), \ldots, u_k^{(t)}$ are functions of t, then

$$D^n[u_1(t)u_2(t), \ldots, u_k(t)]$$

$$= (a_1 + a_2 + \cdots + a_k)^n$$

$$= \sum_{n_1+n_2+n_3+\cdots+n_k=n} \frac{\underline{/n}\, a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}}{\underline{/n_1}\, \underline{/n_2}\, \underline{/n_3} \cdots \underline{/n_k}} \tag{A1}$$

where

$$a_l^0 = u_l(t), \quad a_l^m = D^m u_l(t), \quad D^m = \frac{\mathrm{d}^m}{\mathrm{d}t^m}$$

$$l = 1, 2, \ldots, k, \quad m = 0, 1, 2, \ldots, n. \tag{A2}$$

Proof: we shall give proof of this theorem by using the principle of mathematical induction
For $k = 1$, clearly, we have

$$D^n u_1(t) = a_1^n$$

(which is true), for $k = 2$,

$$D^n\{u_1(t)u_2(t)\} = (a_1 + a_2)^n$$

$$= \sum_{r=0}^{n} \binom{n}{v} a_1^{n-r} a_2^r$$

$$= \sum_{r=0}^{n} \binom{n}{v} \frac{\mathrm{d}u_1^{n-r}(t)}{\mathrm{d}t^{n-r}} \frac{\mathrm{d}^r u_2(t)}{\mathrm{d}t^r}. \tag{A3}$$

Equation (A3) is clearly the statement of Leibnitz's theorem on nth differentiation of a product of two functions, hence the theorem is true for $k = 2$. Let the statement of the above theorem (i.e. eqn (A1), (A2)) be true for $k = m$, then we shall prove that the theorem is true for $k = m + 1$.

To prove this, let us consider:

$$D^n\{u_1(t)u_2(t) \cdots u_m(t)u_{m+1}(t)\}$$

$$= \sum_{r=0}^{n} \binom{n}{v} D^{n-r}\{u_1(t)u_2(t) \cdots u_m(t)\} D^r u_{m+1}^{(t)}$$

(by use of Leibnitz's theorem)

$$= \sum_{r=0}^{n} \binom{n}{v} (a_1 + a_2 + \cdots a_m)^{n-r} a_{m+1}^r.$$

(since the theorem via eqns (A1), (A2) is true for $k = m$)

$$= (a_1 + a_2 + a_3 + \cdots + a_{m+1})^n \tag{A4}$$

(by use of binomial theorem)

$$= \sum_{n_1+n_2+n_3+\cdots+n_{m+1}=n} \frac{\underline{/n}\, a_1^{n_1} a_2^{n_2} \cdots a^{n_{m+1}}}{\underline{/n_1}\, \underline{/n_2}\, \underline{/n_3} \cdots \underline{/m+1}} \tag{A5}$$

(by use of multinomial theorem).

Equations (A4) and (A5) imply that the theorem is true for all K.

This completes the proof of the theorem.